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On a multivariate version of Spearman's correlation coefficient for regression: Properties and Applications

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ABSTRACT
The aim of the present paper is to examine a multivariate extension of the Spearman dependence coefficient which can be deployed in the framework of regression analysis. The coefficient describes how well a response random variable can be approximated by a multivariate monotonously increasing function of a certain number of regressors. We introduce estimators of the dependence coefficient and prove their convergence rate and asymptotic normality.

KEYWORDS
Dependence measures, Spearman's ρ, Spearman's footrule, estimators for dependence measures.

1. Introduction

In this paper we introduce a multivariate version of Spearman's rho tailored for use in regression analysis. This dependence coefficient is introduced as a straightforward extension of the well-known Spearman coefficient of two random variables (cf. Nelsen (2006), pp. 167). It is defined in a similar way to the Kendall regression coefficient which was introduced in Liebscher (2021).

Let Y be the response variable in a regression setup. The regressor variables are combined in a random vector $X = (X_1, \dots, X_k)^T$. We want to contribute to answering the question of how strong the dependence of the response variable Y on the regressor vector X is. The Spearman regression coefficient measures the degree of how well Y can be approximated by a strictly increasing function of X. Hereby the distribution of X is considered as fixed. In the framework of non-linear regression analysis, multiple correlation has not been studied to our knowledge, with exception being the paper by Alfons et al. (2017). This paper deals with bivariate correlations of suitable projections of the regressors.

The main goals of the paper are to study the properties of the Spearman regression coefficient, and to examine an estimator of the coefficient. We introduce a new statistical measure on the basis of copulas. It has the advantage that it does not depend on

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the marginal distributions, and it is robust against outliers. Since the copula is invariant under strictly increasing transformations (see Nelsen (2006), Theorem 2.4.3), this invariance carries over to the dependence coefficient. The coefficient we propose here can be deployed in regression analysis. In case of a high-dimensional vector X of regressors, we are interested in those components which strongly influence the response variable Y. Moreover, after some steps in the analysis, the problem is to find out regressor variables with a strong potential for improving the model, and the residual variable serves as response variable. In both tasks, Spearman's regression coefficient can be a useful tool.

Spearman's rho, Kendall's tau and the Gini measure are classical copula-based coefficients describing the dependence of two random variables (cf. Nelsen (2006), Chapter 5). Multivariate extensions of such measures are investigated in several papers. Joe (1990) introduced in his paper a general class of multivariate dependence measures connected with Kendall's tau and Blomqvist's beta. A multivariate extension of Spearman's rho is examined in Schmid and Schmidt (2007a). Similar approaches in this direction are published by Schmid and Schmidt (2010) (a lot of references to other papers are given there), Koch and De Schepper (2011) and Dhaene et al. (2014). A local dependence measure is considered in Latif and Morettin (2014). In the one-dimensional case ($k = 1$), a coefficient describing the lack of comonotonicity by using a smoothing estimator is studied in Qoyyimi and Zitikis (2015). In Liebscher (2014) the author provides another generalization of Spearman's ρ in the case $k = 1$. The striking difference of the mentioned multivariate dependence measures and our coefficient is that the measures in the mentioned papers describe the deviation from the independence of all components of the random vector or/and the deviation from comonotonicity. Here comonotonicity means that any component is a strictly increasing function of any other component almost surely. We pursue a different aim: the description of the dependence of one response variable on a random vector, and the deviation from the situation where the response variable is a monotone function of the random vector.

The paper is organized as follows: In Section 2 we discuss the classical Spearman coefficient. In Section 3 we introduce the new Spearman regression coefficient which describes the dependence between a response variable and a random vector. We deal with the estimation of this generalized coefficient in Section 4. There we give results on the convergence rate and the asymptotic normality of the sample version of the coefficient. Section 5 contains two extensions of the dependence coefficient. The first one describes the deviation from the situation of a regression function which is increasing in some variables and decreasing in the other variables. The idea of the other extension is to split the regressor domain into sub-cuboids and to describe the dependence on the subsets. The reader finds the proofs in Section 6.

2. Spearman's dependence coefficient of two variables

In this section, we consider the Spearman dependence measure of real random variables X and Y with joint distribution function H (cf. Section 5.1.2 in Nelsen (2006)) F and G are the distributions of X, Y, respectively. It is assumed that F and G are continuous. In view of Sklar's Theorem (see Sklar (1959)), we have

$$H(x, y) = C(F(x), G(y)) \quad \text{for } x, y \in \mathbb{R}.$$

Hereby C is the uniquely determined copula of X, Y. Next we give the definition of Spearman's ρ:

$$\begin{aligned}\rho_S &= 1 - 6 \cdot \int_0^1 \int_0^1 (u-v)^2 \,\mathrm{d}C(u,v) = 12 \int_0^1 \int_0^1 uv \,\mathrm{d}C(u,v) - 3 \\ &= 1 - 6\mathbb{E}(F(X) - G(Y))^2 = 12\mathbb{E}\left(F(X)G(Y)\right) - 3 \\ &= 12 \int_0^1 \int_0^1 C(u,v) \,\mathrm{d}u\mathrm{d}v - 3. \qquad (1)\end{aligned}$$

In Liebscher (2014) ρ_S is generalized by replacing the square by a power or a function fulfilling certain conditions. Let $U = F(X), V = G(Y)$. Obviously, the random variables U and V have a uniform distribution on $[0,1]$, and C is the joint distribution function of (U, V). Observe that

$$\begin{aligned}&\int_0^1 \int_0^1 \mathbb{P}\{U > u, V > v\} \; dudv \\ = &\int_0^1 \int_0^1 \left(\mathbb{P}\{U > u\} + \mathbb{P}\{V > v\} + \mathbb{P}\{U \le u, V \le v\} - 1\right) \; dudv \\ = &\int_0^1 \int_0^1 \mathbb{P}\{U \le u, V \le v\} \; dudv.\end{aligned}$$

We can rewrite the definition formula of ρ_S as follows in a symmetrized version:

$$\rho_S = 6 \int_{[0,1]^2} \mathbb{P}\{U \le u, V \le v\} \,\mathrm{d}u\mathrm{d}v + 6 \int_{[0,1]^2} \mathbb{P}\{U > u, V > v\} \,\mathrm{d}u\mathrm{d}v - 3. \qquad (2)$$

This formula is the starting point for the generalization in the next section.

3. Multivariate Spearman's rho for regression

3.1. Introduction of the coefficient

Let Y be the real response random variable and $X = (X_1, \ldots, X_d)^T$ a random vector containing the regressor variables. We look for a measure describing the dependence of Y on X by a monotonously increasing relationship. The distribution functions of the components of X are denoted by $F_1, \ldots, F_d$, the distribution function of Y is denoted by G. The distribution functions $F_1, \ldots, F_d$, G are assumed to be continuous. H is the joint distribution function of $(X_1, \ldots, X_d, Y)$. We obtain the random vector $U = (U_1, \ldots, U_d)^T$ and the random variable V by transforming the marginal distribution of all components of X and Y into a uniform one:

$$U_i = F_i(X_i), \quad V = G(Y).$$

$C : [0,1]^{d+1} \to [0,1]$ denotes the joint copula of $(X_1, \ldots, X_d, Y)$, i.e. the distribution function of $(U_1, \ldots, U_d, V)$. The copula of vector X alone is denoted by C_X. We consider the joint distribution of X and hence C_X as fixed and given. For $x, y \in \mathbb{R}^d$, $x \le y$ means that $x_i \le y_i$ for all i, similarly for $x < y$. Let C_0 be any strictly increasing

d-dimensional copula. The choice of C_0 will be discussed later. In view of formula (2), we consider the following generalization of Spearman's rho

$$\begin{aligned}\rho_R &= a\int_{[0,1]^{d+1}} \left(\mathbb{P}\{U_1 \le u_1, \dots, U_d \le u_d, V \le v\}\right. \\ &\qquad \left.+\mathbb{P}\{U_1 > u_1, \dots, U_d > u_d, V > v\}\right) \mathrm{d}C_0(u)\mathrm{d}v - b, \qquad (3)\end{aligned}$$

where $u = (u_1, \dots, u_d)^T$. In the case $d = 1, a = 6, b = 3$, ρ_R coincides with Spearman's rho ρ_S, according to (2).

We determine the parameters a and b in (3) such that the following two conditions are fulfilled:

(i) $\rho_R = 0$ provided that X and Y are independent.

(ii)

$$\max_{C:C \text{ copula}, C(u_1,\dots,u_d,1)=C_X(u)\ \forall u\in[0,1]^d} \rho_R(C) = 1.$$

Condition (ii) means that $\rho_R \le 1$ and the maximum of ρ_R can be achieved in a certain situation discussed later.

We introduce $\mathbf{1}()$ to be 1 if the condition in the parentheses is fulfilled and 0 otherwise. The survival copula $\check{C}_0$ of C_0 is given by

$$\check{C}_0(u_1, \dots, u_d) = \bar{C}_0(1 - u_1, \dots, 1 - u_d),$$

where

$$\bar{C}_0(u_1, \dots, u_d) = \int_{[0,1]^{d+1}} \mathbf{1}\left(u_i < \bar{u}_i\ \forall i\right)\ \mathrm{d}C_0(\bar{u}).$$

To shorten formulas, we introduce a test function $\varphi : [0, 1]^d \to [-1, 1]$:

$$\varphi(u) = \bar{C}_0(u) - C_0(u) = \check{C}_0(1 - u_1, \dots, 1 - u_d) - C_0(u).$$

Note that φ is strictly decreasing since C_0 is strictly increasing. Especially for $d = 2$ and 3, we have

$$\begin{aligned}\varphi(u_1, u_2) &= 1 - u_1 - u_2, \qquad (4)\\ \varphi(u_1, u_2, u_3) &= 1 - u_1 - u_2 - u_3 + C_0(u_1, u_2, 1) + C_0(u_1, 1, u_3) \\ &\quad +C_0(1, u_2, u_3) - 2C_0(u_1, u_2, u_3).\end{aligned}$$

Surprisingly, in the case $d = 2$, φ does not depend on C_0. From (3), we can rewrite ρ_R

using the test function

$$\begin{aligned}
\rho_R &= a\int_{[0,1]^{d+1}} \left(\mathbb{P}\left\{U_1 \le u_1,\ldots,U_d \le u_d, V \le v\right\} + \mathbb{P}\left\{U_1 > u_1,\ldots,U_d > u_d\right\}\right. \\
&\qquad \left. -\mathbb{P}\left\{U_1 > u_1,\ldots,U_d > u_d, V \le v\right\}\right) \mathrm{d}C_0(u)\mathrm{d}v - b \\
&= a\int_{[0,1]^{2d+1}} \mathbb{P}\left\{V \le v \mid U = \bar{u}\right\}\left(\mathbf{1}\left(\bar{u}_i \le u_i \ \forall i\right) - \mathbf{1}\left(\bar{u}_i > u_i \ \forall i\right)\right) \mathrm{d}C_0(u)\mathrm{d}C_X(\bar{u})\mathrm{d}v \\
&\quad -b + a\int_{[0,1]^d} \bar{C}_X(u)\mathrm{d}C_0(u) \\
&= a\int_{[0,1]^{d+1}} \mathbb{P}\left\{V \le v \mid U = u\right\}\varphi\left(u\right)\ \mathrm{d}C_X(u)\mathrm{d}v - \tilde{b}, \qquad (5)
\end{aligned}$$

where

$$\tilde{b} = b - a\int_{[0,1]^d} \bar{C}_X(u)\mathrm{d}C_0(u).$$

To ensure condition (ii) above we have to search for a maximum of ρ_R. We denote the distribution function of $-\varphi(U)$ by K:

$$K(w) = \int_{[0,1]^d} \mathbf{1}\left(-\varphi(u) \le w\right)\mathrm{d}C_X(u)$$

for $w \in [-1, 1]$. The following Proposition can be proven for ρ_R of (5):

Proposition 3.1. *Suppose that the measure C_X is absolutely continuous and C_0 is strictly increasing.*
a) The coefficient ρ_R attains its maximum only in the case where $V = K(-\varphi(U))$ a.s. In this case, we have

$$\rho_R = a\mathbb{E}\varphi(U)\left(1 - K(-\varphi\left(U\right))\right) - \tilde{b}.$$

b) The coefficient ρ_R attains its minimum only in the case where $V = 1 - K(-\varphi(U))$ a.s. Then ρ_R is given by

$$\rho_R = a\mathbb{E}\varphi(U)K(-\varphi\left(U\right)) - \tilde{b}.$$

If X and Y are independent, then by (5), we have

$$\begin{aligned}
\rho_R &= a\int_{[0,1]^{d+1}} \varphi(u)\ \mathbb{P}\left\{V \le v\right\}\ \mathrm{d}C_X(u)\ \mathrm{d}v - \tilde{b} \\
&= \frac{a}{2}\mathbb{E}\varphi(U) - \tilde{b}.
\end{aligned}$$

In the following we choose a and $\tilde{b}$ such that the requirements (i) and (ii) are fulfilled:

$$\begin{aligned} \frac{a}{2}\mathbb{E}\varphi(U) &= \tilde{b}, \\ a\mathbb{E}\varphi(U)\left(1 - K(-\varphi(U))\right) - \tilde{b} &= 1 \end{aligned}$$

($\bar{U}$ as in Proposition 3.1). Hence

$$a = \left(\frac{1}{2}\mathbb{E}\varphi(U) - \mathbb{E}\varphi(U)K(-\varphi(U))\right)^{-1} \text{ and } \tilde{b} = \frac{a}{2}\mathbb{E}\varphi(U) \tag{6}$$

hold true. Observe that by partial integration,

$$\int_0^1 \mathbb{P}\{V \le v \mid U = u\}\,\mathrm{d}v = 1 - \mathbb{E}(V \mid U = u),$$

such that (5) implies the final formula for ρ_R

$$\rho_R = \frac{\mathbb{E}\left((1 - \mathbb{E}(V \mid U = u))\varphi(U)\right) - \frac{1}{2}\mathbb{E}\varphi(U)}{\frac{1}{2}\mathbb{E}\varphi(U) - \mathbb{E}\varphi(U)K(-\varphi(U))} = \frac{B}{A}, \tag{7}$$

where $B := \mathbb{E}\varphi(U) - 2\mathbb{E}(V\varphi(U))$, $A := 2a^{-1}$. Further we obtain

$$\begin{aligned} A &= \int_{-1}^{1} z\,(2\,K(z) - 1)\,\mathrm{d}K(z) \\ &= -\int_{[0,1]^d} \varphi(u)\left(2\int_{[0,1]^d} \mathbf{1}\left(-\varphi(\bar{u}) \le -\varphi(u)\right)\,\mathrm{d}C_X(\bar{u}) - 1\right)\mathrm{d}C_X(u) \\ &= \int_{[0,1]^{2d}} \varphi(u)\left(1 - 2\cdot\mathbf{1}\left(\varphi(u) \le \varphi(\bar{u})\right)\right)\,\mathrm{d}C_X(u)\,\mathrm{d}C_X(\bar{u}). \end{aligned} \tag{8}$$

The dependence coefficient is a functional of the copula C since C_X and thus K are fixed. This is to be seen from the formula

$$\rho_R = \rho_R(C) = \frac{1}{A}\left(-\int_{-1}^{1} z\,\mathrm{d}K(z) - 2\int_{[0,1]^{d+1}} v\varphi(u)\,\mathrm{d}C(u_1,\ldots,u_d,v)\right). \tag{9}$$

In the case $d = 1$, the function $\varphi(u) = 1 - 2u$ leads to the equality $\rho_R = \rho_S$. Let us proceed with the two-dimensional case.

3.2. The case $\boldsymbol{d = 2}$

Observe that for the partial Spearman coefficients $\rho_S(X_j, Y)$ of X_j and Y, the equality

$$\rho_S(X_j, Y) = 12\mathbb{E}U_jV - 3$$

holds true ($j = 1, 2$) in view of (1). Let $(\bar{U}_1, \bar{U}_2)$ be an independent copy of (U_1, U_2) having the same distribution function C_X. Note that $\mathbb{E}U_i = \mathbb{E}V = \frac{1}{2}$, and

$\mathbb{P}\left\{U_1 + U_2 \geq \bar{U}_1 + \bar{U}_2\right\} = \frac{1}{2}$. Then by (4),

$$\begin{aligned} \rho_R &= \frac{\mathbb{E}\varphi(U) - 2\mathbb{E}\left(V\varphi(U)\right)}{\mathbb{E}\varphi(U) - 2\mathbb{E}\varphi(U)\mathbf{1}\left\{\varphi\left(U\right) \leq \varphi(\bar{U})\right\}} \\ &= \frac{1 - \mathbb{E}U_1 - \mathbb{E}U_2 - 2\mathbb{E}\left(V(1 - U_1 - U_2)\right)}{1 - \mathbb{E}U_1 - \mathbb{E}U_2 - 2\mathbb{E}(1 - U_1 - U_2)\mathbf{1}\left\{U_1 + U_2 \geq \bar{U}_1 + \bar{U}_2\right\}} \\ &= \frac{\rho_S(X_1, Y) + \rho_S(X_2, Y)}{12\mathbb{E}\left((U_1 + U_2)\,\mathbf{1}\left(U_1 + U_2 \geq \bar{U}_1 + \bar{U}_2\right)\right) - 6}. \end{aligned} \tag{10}$$

$\rho_{X_1,Y}$ and $\rho_{X_2,Y}$ denote Spearman's rho for the pair of variables in the index. In view of (10) coefficient ρ_R is a normed average of pairwise Spearman rho.

To this point the considerations are valid for every strictly increasing copula C_0. Regarding the estimation, the choice $C_0 = C_X$ is not recommendable for two reasons. First, since different C_0 lead to different coefficients we lose the comparability. Moreover, an estimator for C_X has to be plugged in additionally when estimating the coefficient. In the following, we focus on specific cases for C_0.

3.3. The case $C_0 = \Pi$

Here we have

$$\varphi(u) = \prod_{i=1}^{d}(1 - u_i) - \prod_{i=1}^{d} u_i,$$

and in the case $d = 3$,

$$\varphi(u_1, u_2, u_3) = 1 - u_1 - u_2 - u_3 + u_1u_2 + u_1u_3 + u_2u_3 - 2u_1u_2u_3.$$

3.4. The case $C_0 = M$

Let now C_0 be the Fréchet-Hoeffding upper-bound copula $M(u) = \min_i u_i$. Then

$$\varphi(u) = 1 - \max_i u_i - \min_i u_i.$$

3.5. Properties of the coefficient

In the following $C_1 \prec C_2$ denotes a concordance relationship between copulas C_1,C_2 : $[0,1]^2 \to [0,1]$ defined by $C_1(u_1,u_2) \leq C_2(u_1,u_2)$ for $u_1,u_2 \in [0,1]$. The next Theorem 3.2 gives some properties of the dependence measure in the case of general C_0.

Theorem 3.2. *Let C_X be absolutely continuous. Suppose that C_0 is strictly increasing and continuous. The regression coefficient ρ_R has the following properties:*

a) $-1 \leq \rho_R \leq 1$.

b) The identity $\rho_R = 1$ holds iff $V = K(-\varphi(U))$ a.s. which is equivalent to $Y = G^{-1}(K(-\varphi(F_1(X_1),\ldots,F_d(X_d))))$ a.s.. $G^{-1}(K(-\varphi(.)))$ is a function which is monotonously increasing in each argument.

c) If X and Y are independent, then $\rho_R = 0$. If $\rho_R = 0$, then $G(Y)$ and $\varphi(F_1(X_1),\ldots,F_d(X_d))$ are uncorrelated.

d) Let $\Lambda_1, \ldots, \Lambda_{d+1} : \mathbb{R} \to \mathbb{R}$ be strictly increasing functions. The dependence measure of $\Lambda_{d+1}(Y)$ and $(\Lambda_1(X_1), \ldots, \Lambda_d(X_d))$ equals ρ_R.

e) The dependence measure of $-Y$ and X is given by $-\rho_R$.

f) The dependence measure of Y and $-X$ is equal to $-\rho_R$.

g) Let $\{C_n\}$ be a sequence of copulas satisfying $C_n(u_1, \ldots, u_d, 1) = C_X(u)$ and tending pointwise to C. Then $\rho_R(C_n) \to \rho_R(C)$, where $\rho_R(C)$ is defined in (9).

h) Let ρ_{R1} and ρ_{R2} be the regression coefficients for $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$, and $\bar{C}_1, \bar{C}_2 : [0,1]^2 \to [0,1]$ are the resulting copulas of $-\varphi(F_1(X_1), \ldots, F_d(X_d))$ and Y. Assume that $C_{X^{(1)}} = C_{X^{(2)}}$, and $\bar{C}_1 \prec \bar{C}_2$. Then we have

$$\rho_{R1} \leq \rho_{R2}.$$

In Theorem 3.2 we prove properties of the regression coefficient which are similar to those for measures of concordance according to Scarsini (1984). A nice feature is that we can exactly describe the situations for which $\rho_R = \pm 1$ and $\rho_R = 0$ occurs. Assertion b) means that $\rho_R = 1$ occurs exactly in the case where Y is a monotonously increasing function of $X_1, \ldots, X_d$ *a.s.* and this function is defined in terms of G, K, φ, and the F_j's. From parts b) and e) of Theorem 3.2, it follows that $\rho_R = -1 \iff V = 1 - K(-\varphi(U))$ *a.s.* In the case $C_0 = \Pi$, C_0 is strictly increasing and φ is continuously differentiable so that important assumptions of Theorem 3.2 are fulfilled.

4. Estimation of Spearman's regression coefficient

Let $(\bar{X}_1, Y_1), \ldots, (\bar{X}_n, Y_n)$ be a sample of independent random vectors from $\mathbb{R}^{d+1}$ having distribution function H. Here $\bar{X}_i = (X_i^{(1)}, \ldots, X_i^{(d)})^T$. The empirical distribution functions of $X_1, \ldots, X_d$ and Y are denoted by $F_{1n}, \ldots F_{dn}, G_n$. Let $F(x) = (F_1(x_1), \ldots, F_d(x_d))^T$, $\bar{F}_n(x) = (F_{1n}(x_1), \ldots, F_{dn}(x_d))^T$ and $Z_{in} = -\varphi(\bar{F}_n(\bar{X}_i))$. Taking into account (7) and (8), we introduce the estimator for ρ_R:

$$\hat{\rho}_{Rn} = \frac{\hat{B}_n}{\hat{A}_n},$$

where $\hat{A}_n$ and $\hat{B}_n$ are estimators for A and B, respectively:

$$\begin{aligned} \hat{B}_n &= \frac{1}{n} \sum_{i=1}^{n} \varphi(\bar{F}_n(\bar{X}_i)) \left(1 - 2G_n(Y_i)\right), \\ \hat{A}_n &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(\bar{F}_n(\bar{X}_i)) \left(1 - 2 \cdot \mathbf{1}\left\{\varphi(\bar{F}_n(\bar{X}_i)) \leq \varphi(\bar{F}_n(\bar{X}_j))\right\}\right). \end{aligned}$$

Next we look for formulas for $\hat{A}_n$ and $\hat{B}_n$ which are computationally more favourable. We have

$$\begin{aligned} \hat{B}_n &= \frac{1}{n^2} \sum_{i=1}^{n} Z_{in} \sum_{j=1}^{n} \left(2 \cdot \mathbf{1}\{Y_j \leq Y_i\} - 1\right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} Z_{in} \left(2 \cdot R_Y(Y_i) - n\right), \end{aligned}$$

where $R_Y(Y_i)$ is the rank of Y_i among $Y_1, \ldots, Y_n$. Let $Z_{(1)n}, \ldots, Z_{(n)n}$ be the order statistics of $Z_{1n}, \ldots, Z_{nn}$. Further

$$\begin{aligned} \hat{A}_n &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Z_{in} \left(2 \cdot \mathbf{1}\left\{Z_{in} \geq Z_{jn}\right\} - 1\right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} Z_{(i)n} \left(2i - n\right). \end{aligned} \tag{11}$$

The convergence rate of $\hat{\rho}_{Rn}$ is provided in Theorem 4.2 under appropriate conditions. One of them is given now.

Assumption $\mathcal{C}$: Suppose that C_0 is strictly increasing and continuously differentiable, and C_X has a bounded density c_X. For $w = (u_1, \ldots, u_{l-1}, u_{l+1}, \ldots, u_d) \in \mathbb{R}^{d-1}$, $\overset{-1}{\bar{\varphi}}_w$ defines the inverse function of $\bar{\varphi}_w : u_l \rightsquigarrow \varphi(u)$. Let $I_z = \{w \in [0,1]^{d-1} : -\varphi(0, w) \leq z \leq -\varphi(1, w)\}$. Moreover, the identity

$$\sup_{z \in [-1,1]} \int_{I_z} \varphi_l(u_1, \ldots, u_{l-1}, \overset{-1}{\bar{\varphi}}_w(z), u_{l+1}, \ldots, u_d)^{-1} \, \mathrm{d}w < +\infty$$

holds true. Here $\varphi_l : u \rightsquigarrow \frac{\partial \varphi(u)}{\partial u_l}$. ⊡

$\overset{-1}{\bar{\varphi}}_w$ exists since φ is strictly decreasing and continuous. The next Lemma deals with the important case $C_0 = \Pi$.

Lemma 4.1. *Assumption $\mathcal{C}$ is satisfied if c_X is bounded and $C_0 = \Pi$.*

The following theorem provides the convergence rate of the estimator $\hat{\rho}_{Rn}$.

Theorem 4.2. *Suppose that Assumption $\mathcal{C}$ is satisfied. Then*

$$\hat{\rho}_{Rn} - \rho_R = O\left(\sqrt{\frac{\ln \ln n}{n}}\right) \quad a.s.$$

Next we give a result on asymptotic normality of $\hat{\rho}_{Rn}$ under stronger assumptions. This Theorem 4.3 can be used to construct confidence intervals.

Theorem 4.3. *Suppose that Assumption $\mathcal{C}$ is satisfied, and the partial derivatives of C_0 exist and are Hölder-continuous. Then we have*

$$\sqrt{n}\,(\hat{\rho}_{Rn} - \rho_R) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\rho^2),$$

where $\sigma_\rho^2 = 4\mathbb{E}\bar{\Lambda}^2((X_1, Y_1)),\ \Lambda(y) = -\int_y^1 (2K(z) - 1)\,dz,$

$$\begin{aligned}\bar{\Lambda}(x,y) \;=\; & -B\left(\int_{\mathbb{R}^d} \sum_{j=1}^{d} \bar{\varphi}_j(\bar{F}(\bar{x}))\left(\mathbf{1}\{x_j \le \bar{x}_j\} - F_j(\bar{x}_j)\right) dF(\bar{x}) \right. \\ & \left. + \Lambda(-\varphi(\bar{F}(x))) - \int_{-1}^{1} \Lambda(z)\,dK(z)\right) \\ & + A\left(\int_{\mathbb{R}^{d+1}} \sum_{j=1}^{d} \varphi_j(\bar{F}(\bar{x}))\left(\mathbf{1}\{x_j \le \bar{x}_j\} - F_j(\bar{x}_j)\right)(1 - 2G(\bar{y}))\,dH(\bar{x},\bar{y}) \right. \\ & \left. -2\int_{\mathbb{R}^{d+1}} \varphi(\bar{F}(\bar{x}))\left(\mathbf{1}\{y \le \bar{y}\} - G(\bar{y})\right) dH(\bar{x},\bar{y}) + \varphi(\bar{F}(x))(1 - 2G(y)) - B\right).\end{aligned}$$

Here $\bar{\varphi}_1, \ldots, \bar{\varphi}_d$ *are the partial derivatives of* $\Lambda(-\varphi(.))$, $\varphi_1, \ldots, \varphi_d$ *are the partial derivatives of* φ.

For constructing confidence intervals, one needs an estimator for σ_ρ^2. It can be developed by replacing H by H_n, K by K_n, F_j by F_{jn} and G by G_n. The asymptotic normality of the classical Spearman coefficient and some versions of it is shown in Genest et al. (2013). Our Theorem 4.3 represents a similar result.

5. Extended versions of the Spearman regression coefficient

5.1. Versions for increasing and decreasing components

Sometimes one is interested in measuring to which degree the response variable Y depends on a function of the regressors which is increasing in variables $x_i, i \in I$ and decreasing in variables $x_i, i \in \{1, \ldots, d\}\backslash I$. Let $\delta_i = 0$ for $i \in I, \delta_i = 1$ for $i \in \{1, \ldots, d\}\backslash I$, $\delta = (\delta_1, \ldots, \delta_d)$. For any $\delta \in \{0,1\}^d$, the transformed regressor vector is introduced by $\tilde{X}^\delta = ((-1)^{\delta_i} X_i)_{i=1,\ldots,d}$. The coefficient ρ_R of $\tilde{X}^\delta$ and Y according to (7) measures how well the variable Y can be approximated by a function which is increasing in variables $x_i, i \in I$ and decreasing in the remaining variables. We denote this coefficient and its estimator by $\rho_R(Y \mid \tilde{X})$ and $\hat{\rho}_R(Y \mid \tilde{X})$, respectively. The coefficient $\rho_R(Y \mid \tilde{X})$ is an analogous version of $\tau_R(Y \mid \tilde{X})$ from the author's paper (2021).

More specifically, in the case $d = 2$, there are four coefficients, in principle: $\rho_R(Y \mid X_1, X_2), \rho_R(Y \mid -X_1, X_2), \rho_R(Y \mid X_1, -X_2), \rho_R(Y \mid -X_1, -X_2)$ with the relationships (following from Theorem 3.2f))

$$\rho_R(Y \mid -X_1, -X_2) = -\rho_R(Y \mid X_1, X_2), \quad \rho_R(Y \mid -X_1, X_2) = -\rho_R(Y \mid X_1, -X_2).$$

The coefficient $\rho_R(Y \mid X_1, -X_2)$ describes how well the variable Y can be approximated by a function which is increasing in the first component and decreasing in the second one. Other coefficients are interpreted similarly. Overall only two coefficients are to be computed. A comparison of these values shows the most probable direction of ascent of an approximating function.

Example 1: We consider the dataset "Concrete Compressive Strength" from the UCI Machine Learning Repository. The last variable "compressive strength" is the response variable Y, the remaining variables serve as regressors and are denoted by $x_1, \ldots, x_8$. We obtain the following empirical coefficients

$$\hat{\rho}_R(Y \mid X_1, X_2, X_3, -X_4, X_8) = 0.81702,\ \hat{\rho}_R(Y \mid X_1, X_2, -X_3, -X_4, X_8) = 0.78706,$$
$$\hat{\rho}_R(Y \mid X_1, -X_4) = 0.54025,\ \hat{\rho}_R(Y \mid X_1, X_8) = 0.76406.$$

The highest values indicate that there can be found suitable model functions to model the dependence of Y and the regressors. In view of these results it is expected that Y can be well approximated by a function which is increasing in the variables x_1, x_2, x_3, x_8 and decreasing in x_4.

5.2. Splitting the domain

Consider the splitting point $q \in (0,1)^d$ which divides the copula plane $[0,1]^d$ into 2^d subregions. For $\nu \in \{0,1\}^d$, let $I^\nu = \{u \in [0,1]^d : u_i \leq q_i$ for $i : \nu_i = 1,\ u_i > q_i$ for $i : \nu_i = 0\}$. Especially in the case $d = 2$, we obtain $I^{(1,1)} = \{u : u_1 \leq q_1, u_2 \leq q_2\}$, $I^{(1,0)} = \{u : u_1 \leq q_1, u_2 > q_2\}$, $I^{(0,1)} = \{u : u_1 > q_1, u_2 \leq q_2\}$, $I^{(0,0)} = \{u : u_1 > q_1, u_2 > q_2\}$. We introduce the distribution functions $F_i^{\delta,\nu}$ and G^ν of $(-1)^{\delta_i} X_i$ and Y, respectively, given $U \in I^\nu$. Further we define $U^{\delta,\nu} = (F_i^{\delta,\nu}((-1)^{\delta_i} X_i))_{i=1\ldots d}$ and $V^\nu = G^\nu(Y)$ for $\delta, \nu \in \{0,1\}^d$. $K^{\delta,\nu}$ denotes the distribution function of $-\varphi(U^{\delta,\nu})$ given $U \in I^\nu$. On the subregions the denominator A and the numerator B of the dependence coefficient are computed by the formulas

$$\begin{aligned} A^{\delta,\nu} &= \mathbb{E}\left(\varphi(U^{\delta,\nu}) - 2\varphi(U^{\delta,\nu}) K^{\delta,\nu}(-\varphi\left(U^{\delta,\nu}\right)) \mid U \in I^\nu\right), \\ B^{\delta,\nu} &= \mathbb{E}\left((1 - 2V^\nu)\varphi(U^{\delta,\nu}) \mid U \in I^\nu\right) \end{aligned}$$

for $\delta, \nu \in \{0,1\}^d$, which leads to the dependence coefficients

$$\rho_{RS}^{\delta,\nu} = \frac{B^{\delta,\nu}}{A^{\delta,\nu}} \tag{12}$$

for $\delta, \nu \in \{0,1\}^d$. $\rho_{RS}^{\delta,\nu}$ represents the Spearman regression coefficient for direction δ and the data satisfying $U \in I^\nu$. If $\rho_{RS}^{\delta,\nu}$ is close to 1, then it means that given $U \in I^\nu$, Y can be well-approximated by a function which is increasing w.r.t. x_i with $i : \delta_i = 0$, and decreasing w.r.t. x_i with $i : \delta_i = 1$. A multivariate Spearman coefficient on subregions is considered in Schmid and Schmidt (2007b) using similar ideas.

Based on the construction principle in Liebscher (2021), we introduce a general regression coefficient for the split domain

$$\rho_{RS} = \max_{\Delta : \{0,1\}^d \to \{0,1\}^d} \frac{B(\Delta)}{A(\Delta)}, \tag{13}$$

where

$$A(\Delta) = \sum_{\nu \in \{0,1\}^d} \mathbb{P}\{U \in I^\nu\} A^{\Delta(\nu),\nu}, \quad B(\Delta) = \sum_{\nu \in \{0,1\}^d} \mathbb{P}\{U \in I^\nu\} B^{\Delta(\nu),\nu}.$$

The maximum is taken over all functions $\Delta : \{0,1\}^d \to \{0,1\}^d$ assigning a directional vector δ to each subregion. Let $n_\nu = \sum_{i=1}^n \mathbf{1}\left\{\bar{F}_n(X_i) \in I^\nu\right\}$, $\tilde{X}_i^\delta = ((-1)^{\delta_j} X_i^{(j)})_{j=1\dots d}$, and

$$\begin{aligned} G_n^\nu(y) &= \frac{1}{n_\nu} \sum_{i=1}^n \mathbf{1}\left\{Y_i \le y, \bar{F}_n(X_i) \in I^\nu\right\}, \\ F_{nj}^{\delta,\nu}(z) &= \frac{1}{n_\nu} \sum_{i=1}^n \mathbf{1}\left\{(-1)^{\delta_j} X_i^{(j)} \le z, \bar{F}_n(X_i) \in I^\nu\right\} \end{aligned}$$

for $j = 1 \dots d$, $y, z \in \mathbb{R}$. Furthermore, we define $(\bar{F}_n^{\delta,\nu}(x) = (F_{nj}^{\delta,\nu}(x_j))_{j=1\dots d})$

$$\begin{aligned} A_n^{\delta,\nu} &= \frac{1}{n_\nu^2} \sum_{i=1}^n \sum_{k=1}^n \varphi(\bar{F}_n^{\delta,\nu}(\tilde{X}_i^\delta)) \left(1 - 2 \cdot \mathbf{1}\left\{\varphi(\bar{F}_n^{\delta,\nu}(\tilde{X}_i^\delta)) \le \varphi(\bar{F}_n^{\delta,\nu}(\tilde{X}_k^\delta))\right\}\right) \\ &\qquad \mathbf{1}\left\{\bar{F}_n(X_i), \bar{F}_n(X_k) \in I^\nu\right\}, \\ B_n^{\delta,\nu} &= \frac{1}{n_\nu} \sum_{i=1}^n (1 - 2G_n^\nu(Y_i)) \varphi(\bar{F}_n^{\delta,\nu}(\tilde{X}_i^\delta)) \mathbf{1}\left\{\bar{F}_n(X_i) \in I^\nu\right\}. \end{aligned}$$

In a straightforward way, we can establish the estimators for $\rho_{RS}^{\delta,\nu}$ and ρ_{RS}

$$\hat{\rho}_{RS}^{\delta,\nu} = \frac{B_n^{\delta,\nu}}{A_n^{\delta,\nu}}, \qquad \hat{\rho}_{RS} = \max_{\Delta:\{0,1\}^d \to \{0,1\}^d} \frac{B_n(\Delta)}{A_n(\Delta)},$$

where

$$A_n(\Delta) = \sum_{\nu \in \{0,1\}^d} n_\nu A_n^{\Delta(\nu),\nu}, \quad B_n(\Delta) = \sum_{\nu \in \{0,1\}^d} n_\nu B_n^{\Delta(\nu),\nu}$$

In applications, lower dimensional considerations should be preferable with a view to the high complexity of values $\rho_{RS}^{\delta,\nu}$ for large d. For $d = 2$, we can write

$$\begin{aligned} \hat{\rho}_{RS} &= \max_{\Delta(0,0)\dots\Delta(1,1) \in \{0,1\}^2} \left(n_{(1,1)} B_n^{\Delta(1,1),(1,1)} + n_{(0,1)} B_n^{\Delta(0,1),(0,1)}\right. \\ &\qquad \left. + n_{(1,0)} B_n^{\Delta(1,0),(1,0)} + n_{(0,0)} B_n^{\Delta(0,0),(0,0)}\right) \\ &\qquad \left(n_{(1,1)} A_n^{\Delta(1,1),(1,1)} + n_{(0,1)} A_n^{\Delta(0,1),(0,1)}\right. \\ &\qquad \left. + n_{(1,0)} A_n^{\Delta(1,0),(1,0)} + n_{(0,0)} A_n^{\Delta(0,0),(0,0)}\right)^{-1}. \end{aligned}$$

Simultaneously, the directions δ leading to the maximum in (13) can be taken into account. These directions contain the information about the shape of the function behind the data. Example 2 should illustrate this aspect.

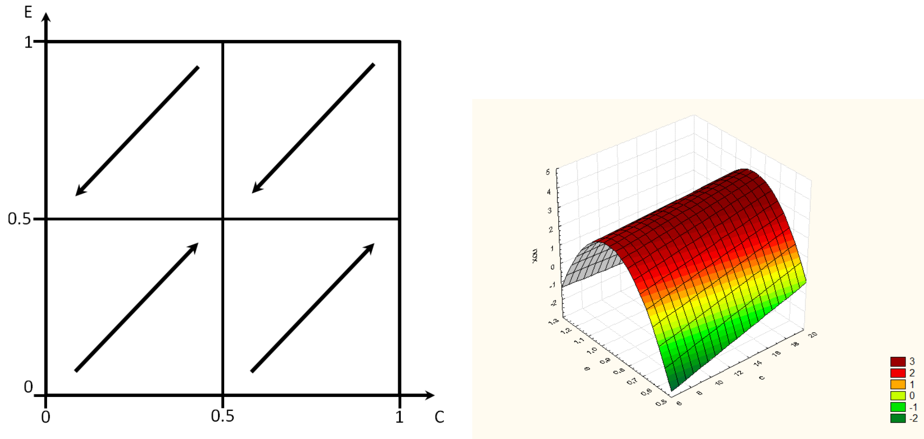

Figure 1. left: directions of maximum dependence (original data), right: fitted quadratic model according to (5.2)

EXAMPLE 2: Dataset "ethanol" from single-cylinder engine study of exhaust emissions by Brinkman (1981). NOX is the response variable indicating the nitrogen oxide exhaust of the engine. C and E are the regressor variables. We choose the splitting point $q_1 = q_2 = 0.5$. Then the regression coefficient (13) is estimated. The results are as follows (for each split region the maximum value is underlined)

$$\begin{aligned}
\hat{\rho}_{RS}^{(0,0),(1,1)} &= \underline{0.82239},\ \hat{\rho}_{RS}^{(0,1),(1,1)} = -0.68636, \\
\hat{\rho}_{RS}^{(0,0),(1,0)} &= -0.79158,\ \hat{\rho}_{RS}^{(0,1),(1,0)} = 0.71671, \Longrightarrow \hat{\rho}_{RS}^{(1,1),(1,0)} = \underline{0.79158}, \\
\hat{\rho}_{RS}^{(0,0),(0,1)} &= \underline{0.75043},\ \hat{\rho}_{RS}^{(0,1),(0,1)} = -0.63966, \\
\hat{\rho}_{RS}^{(0,0),(0,0)} &= -0.74490,\ \hat{\rho}_{RS}^{(0,1),(0,0)} = -0.70009, \Longrightarrow \hat{\rho}_{RS}^{(1,1),(0,0)} = \underline{0.74490} \\
\hat{\rho}_{RS} &= 0.77101.
\end{aligned}$$

For each subregion, the maximum dependence coefficient is underlined. The maximizer for $\hat{\rho}_{RS}$ is given by $(1,1) \rightsquigarrow (0,0), (1,0) \rightsquigarrow (1,1), (0,1) \rightsquigarrow (0,0), (0,0) \rightsquigarrow (1,1)$. The same result is obtained by looking for the most relevant directions (highest value of $\rho_{RS}^{\delta,\nu}$) in the particular subregion. In the left part of Figure 1, we see the expected directions of the slope of the function (according to the mentioned maximizer) in the four split regions.

In the next step we fit the following regression model to the data:

$$Y_i = \beta_1 + \beta_2 c_i + \beta_3 e_i + \beta_4 c_i e_i + \beta_5 e_i^2 + \varepsilon_i$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are independent random variables with zero mean and $\mathbb{V}ar(\varepsilon_i) = \sigma^2$. Using least squares estimation the regression function is fitted. This function is depicted in the right part of Figure 1. We compute the residuals $\hat{\varepsilon}_i = Y_i - \hat{Y}_i$ and analyze the dependence on the regressors:

$$\begin{aligned}
\hat{\rho}_{RS}^{(0,0),(1,1)} &= \underline{0.72029},\ \hat{\rho}_{RS}^{(0,1),(1,1)} = -0.31364, \\
\hat{\rho}_{RS}^{(0,0),(1,0)} &= -0.12421,\ \hat{\rho}_{RS}^{(0,1),(1,0)} = -0.08959, \Longrightarrow \hat{\rho}_{RS}^{(1,1),(1,0)} = \underline{0.12421}, \\
\hat{\rho}_{RS}^{(0,0),(0,1)} &= 0.12791,\ \hat{\rho}_{RS}^{(0,1),(0,1)} = -0.31343, \Longrightarrow \hat{\rho}_{RS}^{(1,0),(0,1)} = \underline{0.31343} \\
\hat{\rho}_{RS}^{(0,0),(0,0)} &= \underline{0.16761},\ \hat{\rho}_{RS}^{(0,1),(0,0)} = 0.08778, \\
\hat{\rho}_{RS} &= 0.31315.
\end{aligned}$$

Here the minimizer for $\hat{\rho}_{RS}$ is given by $(1,1) \rightsquigarrow (0,0), (1,0) \rightsquigarrow (1,1), (0,1) \rightsquigarrow (1,0), (0,0) \rightsquigarrow (0,0)$. We see that there is only low dependence of the residuals on the regressors C and E. Perhaps one can model a dependence only in the region $(1,1)$.

6. Proofs

6.1. Auxiliary statements

In the first lemma we prove an important property of K.

Lemma 6.1. *Assume that the measure C_X is absolutely continuous and C_0 is strictly increasing. Then K is continuous and $K(-\varphi(U))$ has a continuous uniform distribution on $[0,1]$.*

PROOF: Let (w_n) be a sequence of real numbers with $w_n \uparrow w, w_n < w$. Applying the dominated convergence theorem, we obtain

$$K(w) - K(w_n) = \int_{[0,1]^d} \mathbf{1}\left(w_n < -\varphi(u) \leq w\right) \mathrm{d}C_X(u) \to \int_{[0,1]^d} \mathbf{1}\left(-\varphi(u) = w\right) \mathrm{d}C_X(u)$$

for $n \to \infty$. The integral on the right hand side gives value 0 since $-\varphi$ is strictly increasing and C_X is absolutely continuous. Hence K is continuous. The lemma is now a consequence of Proposition 2(1) in Embrechts and Hofert (2013). ⊡

In Lemma 6.2, we prove a consequence of Assumption $\mathcal{C}$. Then the validity of Lemma 4.1 is shown.

Lemma 6.2. *Let Assumption $\mathcal{C}$ be fulfilled. Then K has a bounded density.*

PROOF: Without loss of generality, let $l = 1$. For $w \in [0,1]^{d-1}$, $\bar{\varphi}_w^{-1}$ and $\varphi_{u_1}(.,w)$ denote the inverse function and the derivative of $\varphi(.,w)$. First we show that $\varphi_{u_1}(.,w)$ is bounded from below. Let $T = (T_1, \ldots, T_d)^T$ be a random vector having distribution function C_0, and $\tilde{T} = (T_2, \ldots, T_d)^T$. Then

$$\begin{aligned}
\varphi_{u_1}(u) &= \frac{\partial}{\partial u_1} \left(\mathbb{P}\{T > u\} - \mathbb{P}\{T \le u\} \right) \\
&= \frac{\partial}{\partial u_1} \left(\int_{1-u_1}^{1} \mathbb{P}\left\{ \tilde{T} > u \mid T_1 = z \right\} dz - \int_0^{u_1} \mathbb{P}\left\{ \tilde{T} \le u \mid T_1 = z \right\} dz \right) \\
&= -\mathbb{P}\left\{ \tilde{T} > \tilde{u} \mid T_1 = u_1 \right\} - \mathbb{P}\left\{ \tilde{T} \le \tilde{u} \mid T_1 = u_1 \right\} \qquad (14)
\end{aligned}$$

holds for *a.e.* $u \in [0,1]^d, \tilde{u} = (u_2, \ldots, u_d)^T$. Observe that φ is strictly decreasing. Let $\tilde{X} = (X_2, \ldots, X_d)^T, \tilde{U} = (U_2, \ldots, U_d)^T$. For the density f_Z of K, we obtain

$$f_Z(z) \le \sup_{u \in [0,1]^d} c_X(u) \int_{R_z} \left(-\varphi_{u_1}(\overset{-1}{\bar{\varphi}}_w(-z), w) \right)^{-1} \mathrm{d}w$$

for $z \in [-1, 1]$, which is bounded by assumption $\mathcal{C}$. ⊡

Proof of Lemma 4.1: Without loss of generality, let $l = 1$. Suppose that $C_0 = \Pi$. In view of (14), we have then

$$\varphi_{u_1}(u) = -\mathbb{P}\left\{ \tilde{T} > \tilde{u} \right\} - \mathbb{P}\left\{ \tilde{T} \le \tilde{u} \right\} = -\left(\prod_{j=2}^{d} (1 - u_j) + \prod_{j=2}^{d} u_j \right)$$

for $u \in [0,1]^d$. The maximum of the integral $I(z) = \int_{R_z} \left(-\varphi_{u_1}(\overset{-1}{\bar{\varphi}}_w(-z), w) \right)^{-1} \mathrm{d}w$ is achieved for $z = 0$ since $R_z \subset R_0$. We now show that $I(0) < +\infty$. In the integral $I(0)$, the range R_z of integration can be divided into subranges $\bar{R}_2 \times \bar{R}_3 \times \ldots \times \bar{R}_d$, where $\bar{R}_j \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. Obviously, $-\varphi_{u_1}(\overset{-1}{\bar{\varphi}}_w(-z), w)$ is bounded from below by a positive number on $[0, \frac{1}{2}]^{d-1}$ and $[\frac{1}{2}, 1]^{d-1}$, and the integral over these subranges is finite. For $d = 2$, the proof is already complete here. Let $d \ge 3$, $J_1 = \{j \ge 4 : \bar{R}_j = [0, \frac{1}{2}]\}$, and $J_2 = \{j \ge 4 : \bar{R}_j = [\frac{1}{2}, 1]\}$. Now we consider one representative of the remaining subranges $[0, \frac{1}{2}] \times [\frac{1}{2}, 1] \times \bar{R}_3 \times \ldots \times \bar{R}_d$, and the corresponding part of integral I:

$$\tilde{I} = \int_{[0,\frac{1}{2}] \times [\frac{1}{2},1] \times \bar{R}_4 \times \ldots \times \bar{R}_d} \left(\prod_{j=2}^{d} u_j + \prod_{j=2}^{d} (1 - u_j) \right)^{-1} du_2 du_3 d\tilde{u}^{(d-3)},$$

where $\tilde{u}^{(d-3)} = (u_4, \ldots, u_d)^T$. We deduce

$$\begin{aligned}
\tilde{I} &\leq 2^{d-3} \int_{[0,\frac{1}{2}]\times[\frac{1}{2},1]\times\bar{R}_4\times\ldots\times\bar{R}_d} \left(u_2 \prod_{j\in J_1} u_j + (1-u_3) \prod_{j\in J_2} (1-u_j) \right)^{-1} du_2 du_3 d\tilde{u}^{(d-3)} \\
&= 2^{d-3} \int_{[0,1/2]^{d-1}} \left(u_2 \prod_{j\in J_1} u_j + u_3 \prod_{j\in J_2} u_j \right)^{-1} du_2 du_3 d\tilde{u}^{(d-3)} \\
&= 2^{d-3} \int_{[0,1/2]^{d-3}} \left(\prod_{j\geq 4} u_j \right)^{-1} \int_0^{2^{-1}\prod_{j\in J_2} u_j} \int_0^{2^{-1}\prod_{j\in J_1} u_j} (u_2+u_3)^{-1} du_2 du_3 d\tilde{u}^{(d-3)} \\
&\leq 2^{d-2} \int_{[0,1/2]^{d-3}} \left(\prod_{j\geq 4} u_j \right)^{-1} \left(2^{-1} \prod_{j\in J_1} u_j \right)^{1/2} \left(2^{-1} \prod_{j\in J_2} u_j \right)^{1/2} d\tilde{u}^{(d-3)} \\
&= 2^{d-3} \int_{[0,1/2]^{d-3}} \left(\prod_{j\geq 4} u_j \right)^{-1/2} d\tilde{u}^{(d-3)} < +\infty
\end{aligned}$$

since

$$\begin{aligned}
\int_0^a \int_0^b (u_2+u_3)^{-1} du_2 du_3 &= a \ln\left(1+\frac{b}{a}\right) + b\ln\left(1+\frac{a}{b}\right) \\
&\leq 2\sqrt{ab}
\end{aligned}$$

The parts of I over other subranges can be handled similarly. Hence $I(0) < +\infty$ follows which completes the proof. ⊡

The following two lemmas will be used in the proof of Proposition 3.1. Subsequently, we analyse extremal property of the functional

$$I_\psi := \int_{[0,1]^d} \varphi(u)\, \psi(u) \mathrm{d}C_X(u),$$

where $\psi : [0,1]^d \to \mathbb{R}$ is a measurable function. Function φ is introduced in Section 3.1.

Lemma 6.3. *Let $v \in (0,1)$ be given, and Ψ be the set of measurable functions $\psi : [0,1]^d \to \mathbb{R}$ with $0 \leq \psi(u) \leq 1$ and*

$$\int_{[0,1]^d} \psi(u) dC_X(u) = v. \tag{15}$$

Assume that the measure C_X is absolutely continuous and C_0 is strictly increasing. Then I_ψ attains its maximum on the set Ψ at ψ_0, where $\psi_0(u) = \mathbf{1}(K(-\varphi(u)) \leq v)$ for $u \in [0,1]^d$. Moreover,

$$I_\psi < I_{\psi_0}$$

holds for all functions $\psi \in \Psi$ which differ from ψ_0 on a set of positive C_X-measure.

Proof: Function ψ_0 fulfils condition (15) since by Lemma 6.1,

$$\int_{[0,1]^d} \psi_0(u)\mathrm{d}C_X(u) = \mathbb{P}\{K(-\varphi(U)) \leq v\} = v.$$

Let $J := \{u \in [0,1]^d : K(-\varphi(u)) < v\}$, $\tilde{J} := \{u \in [0,1]^d : K(-\varphi(u)) > v\}$, and $\bar{J} := \{u \in [0,1]^d : K(-\varphi(u)) = v\}$. We consider an arbitrary function $\psi \in \Psi$ which differs from ψ_0 on a set $D \subset [0,1]^d$ of positive C_X-measure. Let $\Delta(u) := \psi(u) - \psi_0(u)$. Therefore by $0 \leq \psi(u) \leq 1$ and (15), function Δ satisfies $D = \{u : \Delta(u) \neq 0\}$, $\Delta(u) < 0$ for $u \in J \cap D$, $\Delta(u) > 0$ for $u \in \tilde{J} \cap D$, and

$$\int_{[0,1]^d} \Delta(u)\mathrm{d}C_X(u) = 0. \tag{16}$$

We introduce the generalized inverse $K^{(-1)}(v) = \inf\{z_0 : K(z_0) \geq v\}$. For $u \in J$, inequality $\varphi(u) + K^{(-1)}(v) > 0$ holds. For $u \in \tilde{J}$, inequality $\varphi(u) + K^{(-1)}(v) \leq 0$ is satisfied. By Lemma 6.1, $\bar{J}$ has zero C_X-measure. Since φ is strictly decreasing and the C_X-measure is absolutely continuous, the set $\{u : \varphi(u) + K^{(-1)}(v) = 0\}$ has zero C_X-measure. Hence $\left(\varphi(u) + K^{(-1)}(v)\right)\Delta(u) < 0$ holds C_X-almost all $u \in (J \cup \tilde{J}) \cap D$. Moreover by (16), we obtain

$$\begin{aligned} I_\psi &= I_{\psi_0+\Delta} = I_{\psi_0} + \int_{[0,1]^d} \varphi(u)\,\Delta(u)\mathrm{d}C_X(u) \\ &= I_{\psi_0} + \int_{[0,1]^d} \Big(\varphi(u) + K^{(-1)}(v)\Big)\,\Delta(u)\ \mathrm{d}C_X(u) \\ &= I_{\psi_0} + \int_{(J\cup\tilde{J})\cap D} \Big(\varphi(u) + K^{(-1)}(v)\Big)\,\Delta(u)\ \mathrm{d}C_X(u) < I_{\psi_0} \end{aligned}$$

which proves the lemma. ⊡

Lemma 6.4. *For two copulas $\bar{C}_1, \bar{C}_2 : [0,1]^2 \to [0,1]$ satisfying $\bar{C}_1 \prec \bar{C}_2$, we have*

$$\int_{[-1,1]\times[0,1]} z\,(1-2v)\ d\bar{C}_1(K(z), v) \geq \int_{[-1,1]\times[0,1]} z\,(1-2v)\ d\bar{C}_2(K(z), v).$$

Proof: Define

$$\eta = \int_{[-1,1]\times[0,1]} (1-z)\ \mathrm{d}\bar{C}_j(K(z), v) = 1 - \int_{-1}^{1} z\ \mathrm{d}K(z).$$

Since $\mathbb{E}(1-2V)=0$ holds true, we obtain

$$\begin{aligned}
&\int_{[-1,1]\times[0,1]} z\,(1-2v)\,\mathrm{d}\bar{C}_j(K(z),v)\\
=\ &\int_{[-1,1]\times[0,1]} (-1+z)\,(1-2v)\,\mathrm{d}\bar{C}_j(K(z),v)\\
=\ &-2\int_{[-1,1]\times[0,1]} (1-z)\,(1-v)\,\mathrm{d}\bar{C}_j(K(z),v)+\eta\\
=\ &-2\int_{[-1,1]\times[0,1]} \left(\int_v^1\int_z^1 d\bar{z}d\bar{v}\right)\mathrm{d}\bar{C}_j(K(z),v)+\eta\\
=\ &-2\int_{[-1,1]\times[0,1]} \bar{C}_j(K(z),v)\mathrm{d}z\mathrm{d}v+\eta \qquad (17)
\end{aligned}$$

by integrating by parts. Since $\bar{C}_1 \prec \bar{C}_2 \Leftrightarrow \bar{C}_1(u)\le \bar{C}_2(u)$ holds for $u\in[0,1]^2$, the lemma follows immediately from (17). ⊡

The following lemma is used several times in proofs of convergence rates concerning the estimator of ρ_R. Here F_{jn} is the empirical distribution function of X_j, and $\bar{F}_n(x)=(F_{1n}(x),\ldots,F_{dn}(x))^T$ for $x\in\mathbb{R}^d$.

Lemma 6.5. *Assume that $F_1,\ldots,F_d$ are continuous.*
a) Then

$$\begin{aligned}
\max_{j=1\ldots d}\sup_{x\in\mathbb{R}^d}|F_{jn}(x)-F_j(x)| &\le \kappa_1\sqrt{\frac{\ln\ln n}{n}}\ \text{a.s.},\\
\sup_{y\in\mathbb{R}}|G_n(y)-G(y)| &\le \kappa_1\sqrt{\frac{\ln\ln n}{n}}\ \text{a.s.}
\end{aligned}$$

for $n\ge n_0(\omega)$ with a constant $\kappa_1>\frac{1}{2}\sqrt{2}$.
b) Then

$$\sup_{x\in\mathbb{R}^d}\left|\varphi(\bar{F}_n(x))-\varphi(\bar{F}(x))\right|\le\kappa_2\sqrt{\frac{\ln\ln n}{n}}\ \text{a.s.}$$

for $x\in\mathbb{R}^d, n\ge n_1(\omega)$, where $\kappa_2>d2^d\kappa_1$ is a constant.

Proof: Assertion a) follows from the law of iterated logarithm for the empirical process (cf. Van der Vaart (1998), p. 268, for example).
b) Note that

$$\sup_{u,\bar{u}\in[0,1]^d}|\varphi(u)-\varphi(\bar{u})|\le(2^d-1)\sum_{j=1}^d|u_j-\bar{u}_j|\,.$$

Apply the inequality of part a) to obtain assertion b). ⊡

6.2. Proofs of the main statements

Proof of Proposition 3.1: Define the conditional distribution function of V given $u \in [0,1]$ by

$$F_{V|U=u}(v) = \mathbb{P}\{V \le v \mid U = u\}$$

for $v \in [0,1]$. Note that

$$\int_{[0,1]^d} F_{V|U=u}(v)\ \mathrm{d}C_X(u) = \mathbb{P}\{V \le v\} = v$$

for $v \in [0,1]$. Let us fix $v \in [0,1]$. Applying Lemma 6.3 we obtain that for every $v \in (0,1)$, the maximum of

$$\int_{[0,1]^d} \varphi(u)\ F_{V|U=u}(v)\ \mathrm{d}C_X(u)$$

is attained in the case where

$$F_{V|U=u}(v) = \mathbf{1}\left(K(-\varphi(u)) \le v\right) \quad \text{for } u \in [0,1] \tag{18}$$

or for a function $u \rightsquigarrow F_{V|U=u}(v)$ differing from (18) on a set of zero C_X-measure. This implies $V = K(-\varphi(U))$ *a.s.* Next we compute the maximum value ρ_R^* of ρ_R in this case using (5):

$$\begin{aligned}\rho_R^* &= a \int_{[0,1]^{d+1}} \varphi(u)\, \mathbf{1}\left(K(-\varphi(u)) \le v\right)\ \mathrm{d}C_X(u)\ \mathrm{d}v - \tilde{b}\\ &= a \int_{[0,1]^d} \varphi(u)\left(1 - K(-\varphi(u))\right)\ \mathrm{d}C_X(u) - \tilde{b}.\end{aligned}$$

This leads to assertion a). Now we consider the minimum value ρ_R^{**} of ρ_R and first that of $\bar{\rho}(v, F_{V|U=u})$ for any fixed $v \in (0,1)$. Observe that for $v \in [0,1]$,

$$\begin{aligned}&\int_{[0,1]^d} \varphi(u)\ F_{V|U=u}(v)\ \mathrm{d}C_X(u)\\ =\ & \mathbb{E}\varphi(U) - \int_{[0,1]^d} \varphi(u)\ \mathbb{P}\{1 - V < 1 - v \mid U = u\}\ \mathrm{d}C_X(u)\\ =\ & \mathbb{E}\varphi(U) - \int_{[0,1]^d} \varphi(u)\ F_{1-V|U=u}(1-v)\ \mathrm{d}C_X(u).\end{aligned} \tag{19}$$

Moreover, we have

$$\int_{[0,1]^d} F_{1-V|U=u}(1-v)\ \mathrm{d}C_X(u) = \mathbb{P}\{V > v\} = 1 - v.$$

An application of Lemma 6.3 yields that for $v \in [0,1]$, the integral (19) reaches its minimum at

$$F_{1-V|U=u}(1-v) = \mathbf{1}\left(K(-\varphi(u)) \le 1-v\right) \quad \text{for } u \in [0,1] \tag{20}$$

or for a function $u \rightsquigarrow F_{1-V|U=u}(1-v)$ differing from (20) on a set of zero C_X-measure. This, in turn, is equivalent to $V = 1 - K(-\varphi(U))$ *a.s.* The minimum value ρ_R^{**} of ρ_R is given by

$$\begin{aligned}
\rho_R^{**} &= a\left(\mathbb{E}\varphi(U) - \int_{[0,1]^{d+1}} \varphi(u)\, \mathbf{1}\left(K(-\varphi(u)) \le 1-v\right)\, \mathrm{d}C_X(u)\mathrm{d}v\right) - \tilde{b} \\
&= a\left(\mathbb{E}\varphi(U) - \int_{[0,1]^{d}} \varphi(u)\, \left(1 - K(-\varphi(u))\right)\, \mathrm{d}C_X(u)\right) - \tilde{b}.
\end{aligned}$$

This proves the Proposition. ⊡

Proof of Theorem 3.2: Proposition 3.1 and (6) imply assertions a) and b). In view of (6), $\rho_R = 0$ holds for independent Y and X. If $\rho_R = 0$ is satisfied, then $B = 0$

$$0 = \mathbb{E}\left((1-2V)\varphi(U)\right) = -2\left(\mathbb{E}V\varphi(U) - \mathbb{E}V\mathbb{E}\varphi(U)\right)$$

showing the validity of c).

d) Since $(\Lambda_1(X_1), \ldots, \Lambda_d(X_d), \Lambda_{d+1}(Y))$ has the same copula C as $(X_1, \ldots, X_d, Y)$ and ρ_R is based on this copula, the assertion d) follows immediately.

e) For the dependence measure $\tilde{\rho}_R$ of $-Y$ and X, we obtain

$$\tilde{\rho}_R = \frac{\mathbb{E}\varphi(U) - 2\mathbb{E}\left((1-V)\varphi(U)\right)}{\mathbb{E}\varphi(U) - 2\mathbb{E}\varphi(U)K(-\varphi(U))} = -\rho_R.$$

f) Let $\tilde{U} = (1-U_1, \ldots, 1-U_d)^T$. Observe that

$$\varphi(\tilde{U}) = \bar{C}_0(\tilde{U}) - C_0(\tilde{U}) = C_0(U) - \bar{C}_0(U) = -\varphi(U).$$

For the dependence measure $\breve{\rho}_R$ of Y and $-X$, we have

$$\begin{aligned}
\breve{\rho}_R &= \frac{-\mathbb{E}\varphi(U) + 2\mathbb{E}\left(V\varphi(U)\right)}{-\mathbb{E}\varphi(U) + 2\mathbb{E}\varphi(U)\mathbf{1}\left\{\varphi(U) \ge \varphi(\bar{U})\right\}} \\
&= \frac{-\mathbb{E}\varphi(U) + 2\mathbb{E}\left(V\varphi(U)\right)}{\mathbb{E}\varphi(U) - 2\mathbb{E}\varphi(U)\mathbf{1}\left\{\varphi(U) < \varphi(\bar{U})\right\}} = -\rho_R.
\end{aligned}$$

g) This part is a consequence of the Portmanteau theorem (see e.g. Van der Vaart (1998), p. 6).

h) If $\tilde{C}$ is the copula of $-\varphi(U)$ and V, then we have

$$\rho_R = -a \cdot \int z\,(1-2v)\,\mathrm{d}\tilde{C}(K(z), v),$$

where $a > 0$. An application of Lemma 6.4 leads to assertion h). ⊡

Next we prove the asymptotic normality of Spearman's rho. For this purpose we analyse first the asymptotics of A_n. Let $Z_{in} = -\varphi(\bar{F}_n(\bar{X}_i)), Z_i = -\varphi(\bar{F}(\bar{X}_i))$. We introduce

$$K_n(z) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Z_{in} \le z\right\}, \quad \bar{K}_n(z) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Z_i \le z\right\}$$

for $z \in [-1, 1]$. The next lemma provides the convergence rate of $K_n - K$.

Lemma 6.6. *Assume that K has a bounded density. Then*

$$\sup_{z\in[-1,1]} |K_n(z) - K(z)| = O\left(\sqrt{\frac{\ln n}{n}}\right) \quad a.s.$$

Proof: From the law of iterated logarithm for the empirical process (cf. Van der Vaart (1998), p. 268), we conclude

$$\sup_{z\in[-1,1]} \left|\bar{K}_n(z) - K(z)\right| = O\left(\sqrt{\frac{\ln\ln n}{n}}\right). \tag{21}$$

Note that

$$K_n(z) - \bar{K}_n(z) = \frac{1}{n}\sum_{i=1}^{n} \left(\mathbf{1}\left\{Z_{in} \le z\right\} - \mathbf{1}\left\{Z_i \le z\right\}\right) = K_{1n}(z) - K_{2n}(z)$$

for $z \in [-1, 1]$, where

$$\begin{aligned} K_{1n}(z) &= \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Z_{in} \le z, Z_i > z\right\}, \\ K_{2n}(z) &= \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Z_i \le z, Z_{in} > z\right\}. \end{aligned}$$

Let $\gamma_n = \kappa_2\sqrt{\frac{\ln\ln n}{n}}$. By Lemma 6.5b),

$$\tilde{Z}_{in} := \varphi(\bar{F}_n(\bar{X}_i)) - \varphi(\bar{F}(\bar{X}_i)) \ge -\gamma_n.$$

Furthermore by (21), we obtain

$$\begin{aligned} K_{1n}(z) &= \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{Z_i + \tilde{Z}_{in} \le z, Z_i > z\right\} \\ &\le \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{z < Z_i \le z + \gamma_n\right\} \\ &= \bar{K}_n(z+\gamma_n) - \bar{K}_n(z) \\ &\le O\left(\sqrt{\frac{\ln\ln n}{n}}\right) + \mathbb{P}\left\{z < Z_i \le z + \gamma_n\right\}. \qquad (22) \end{aligned}$$

On the other hand, we have

$$\sup_{z\in[-1,1]} \mathbb{P}\left\{z < Z_i \le z+\gamma_n\right\} = \sup_{z\in[-1,1]} \left|\int_z^{z+\gamma_n} \psi(t)\mathrm{d}t\right| = O(\gamma_n) = O(\sqrt{\frac{\ln\ln n}{n}}) \qquad (23)$$

since the density ψ of Z_i is bounded. The identities (22) and (23) imply

$$\sup_{z\in[-1,1]} K_{1n}(z) = O\left(\sqrt{\frac{\ln\ln n}{n}}\right).$$

Analogously, one proves that

$$\sup_{z\in[-1,1]} K_{2n}(z) = O\left(\sqrt{\frac{\ln\ln n}{n}}\right).$$

The previous two identities together with (21) complete the proof. ⊡

Observe that by (11),

$$\begin{aligned} \hat{A}_n &= \frac{1}{n}\sum_{i=1}^{n} Z_{in}\left(\frac{2}{n}\sum_{j=1}^{n}\mathbf{1}\{Z_{in} \ge Z_{jn}\} - 1\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} Z_{in}\left(2K_n(Z_{in}) - 1\right). \qquad (24) \end{aligned}$$

We introduce $\Lambda(y) = -\int_y^1 (2K(z)-1)\mathrm{d}z$. The next lemma gives an asymptotic representation of $\hat{A}_n$.

Lemma 6.7. *Assume that Z_i has a bounded density.*
a) Then

$$\hat{A}_n - A = O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \ a.s.$$

b) Suppose in addition that the partial derivatives φ_j of φ are Hölder-continuous. We obtain

$$\hat{A}_n - A = \bar{A}_n + \breve{A}_n + o(n^{-1/2}) \quad a.s.,$$

where the partial derivatives of $\Lambda(-\varphi(.))$ are denoted by $\bar{\varphi}_1, \ldots, \bar{\varphi}_d$, and

$$\begin{aligned} \bar{A}_n &= \frac{1}{n}\sum_{i=1}^{n} \Lambda(Z_i) - \int_{-1}^{1} \Lambda(z)\, dK(z), \\ \breve{A}_n &= \frac{1}{n}\sum_{i=1}^{n} \left(\sum_{j=1}^{d} \bar{\varphi}_j(\bar{F}(\bar{X}_i)) \left(F_{jn}(X_i^{(j)}) - F_j(X_i^{(j)}) \right) \right). \end{aligned}$$

Proof: In view of (8) and (24), we have

$$\begin{aligned} \hat{A}_n - A &= \frac{1}{n}\sum_{i=1}^{n} Z_{in}\left(2K_n(Z_{in}) - 1\right) - \int_{-1}^{1} z(2K(z)-1)dK(z) \\ &= 2A_{1n} + A_{2n}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} A_{1n} &= \frac{1}{n}\sum_{i=1}^{n} Z_{in}\left(K_n(Z_{in}) - K(Z_{in})\right) - \int_{-1}^{1} z(K_n(z) - K(z))\mathrm{d}K(z), \\ A_{2n} &= \int_{-1}^{1} z(2K(z)-1)\mathrm{d}(K_n(z) - K(z)) + 2\int_{-1}^{1} z(K_n(z) - K(z))\mathrm{d}K(z). \end{aligned}$$

Let $Z_{(i)n}$ be the i-th order statistic of $Z_{n1}, \ldots, Z_{nn}$. We deduce

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} Z_{in}K_n(Z_{in}) &= \frac{1}{n}\sum_{i=1}^{n} Z_{(i)n}K_n(Z_{(i)n}) = \frac{1}{n}\sum_{i=1}^{n} Z_{(i)n}\cdot\frac{i}{n} \\ &= \frac{1}{2}\sum_{i=1}^{n} Z_{(i)n}\left(K_n^2(Z_{(i)n}) - K_n^2(Z_{n(i)} - 0)\right) + \frac{1}{2n^2}\sum_{i=1}^{n} Z_{(i)n} \\ &= \frac{1}{2}\int_{-1}^{1} z\ \mathrm{d}K_n^2(z) + O(n^{-1})\ a.s. \end{aligned}$$

Further, we obtain

$$\begin{aligned} A_{1n} &= \frac{1}{2}\int_{-1}^{1} z\ \mathrm{d}K_n^2(z) - \int_{-1}^{1} zK(z)\ \mathrm{d}K_n(z) - \int_{-1}^{1} z(K_n(z)-K(z))\mathrm{d}K(z) + O(n^{-1}) \\ &= \frac{1}{2}\Bigg(\int_{-1}^{1} z\ \mathrm{d}K_n^2(z) - 2\int_{-1}^{1} zK(z)\ \mathrm{d}K_n(z) - 2\int_{-1}^{1} zK_n(z)\ \mathrm{d}K(z) \\ &\quad +2\int_{-1}^{1} zK(z)\ \mathrm{d}K(z)\Bigg) + O(n^{-1}) \\ &= \frac{1}{2}\int_{-1}^{1} z\ \mathrm{d}_z(K_n(z)-K(z))^2 + O(n^{-1}) \\ &= -\frac{1}{2}\int_{-1}^{1} (K_n(z)-K(z))^2\mathrm{d}z = O(\ln\ln n\ n^{-1}) \quad a.s. \end{aligned} \tag{26}$$

by integration by parts and by applying Lemma 6.6. Next we analyze the asymptotics of A_{2n}. Note that $\Lambda'(y) = 2K(z)-1, \Lambda(1) = 0$. By integration by parts, we derive

$$\begin{aligned} A_{2n} &= \int_{-1}^{1} z\ \mathrm{d}_z\left((K_n(z)-K(z))(2K(z)-1)\right) \\ &= -\int_{-1}^{1} (K_n(z)-K(z))(2K(z)-1)\mathrm{d}z \\ &= -\frac{1}{n}\sum_{i=1}^{n}\int_{-1}^{1} \mathbf{1}\left\{Z_{in}\le z\right\}(2K(z)-1)\mathrm{d}z + \int_{-1}^{1} K(z)(2K(z)-1)\mathrm{d}z \\ &= -\frac{1}{n}\sum_{i=1}^{n}\int_{Z_{in}}^{1}(2K(z)-1)\mathrm{d}z - \int_{-1}^{1}\Lambda(z)\mathrm{d}K(z) \\ &= \frac{1}{n}\sum_{i=1}^{n}\Lambda(Z_{in}) - \int_{-1}^{1}\Lambda(z)\mathrm{d}K(z). \end{aligned} \tag{27}$$

a) An application of the law of iterated logarithm leads to

$$\bar{A}_n := \frac{1}{n}\sum_{i=1}^{n}\Lambda(Z_i) - \int_{-1}^{1}\Lambda(z)\mathrm{d}K(z) = O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \quad a.s. \tag{28}$$

Note that Λ has the bounded derivative $2K(.)-1$, and φ is Lipschitz continuous (φ is constructed from a copula C_0). Further by Lemma 6.5, we have

$$\begin{aligned} \left|A_{2n} - \bar{A}_n\right| &\le \frac{1}{n}\sum_{i=1}^{n}\left|\Lambda(-\varphi(\bar{F}_n(\bar{X}_i))) - \Lambda(-\varphi(\bar{F}(\bar{X}_i)))\right| \\ &\le \sup_{x\in\mathbb{R}^d}\left|\varphi(\bar{F}_n(x)) - \varphi(\bar{F}(x))\right| = O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \quad a.s. \end{aligned}$$

Assertion a) is now a consequence of (25), (26) and (28).

b) Since K is Lipschitz and functions $\bar{\varphi}_1, \ldots, \bar{\varphi}_d$ are Hölder-continuous with exponent $\alpha \in (0,1]$, we derive

$$\begin{aligned} A_{2n} - \bar{A}_n &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda(-\varphi(\bar{F}_n(\bar{X}_i))) - \Lambda(-\varphi(\bar{F}(\bar{X}_i)))\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{d}\bar{\varphi}_j(\bar{F}(\bar{X}_i))\left(F_{jn}(X_i^{(j)}) - F_j(X_i^{(j)})\right)\right) + R_n \\ &= \breve{A}_n + R_n, \end{aligned} \tag{29}$$

from (27) and by Lemma 6.5,

$$\begin{aligned} |R_n| &\leq C_2 \cdot \frac{1}{n}\sum_{i=1}^{n}\left|F_{jn}(X_i^{(j)}) - F_j(X_i^{(j)})\right|^{1+\alpha} \\ &= O\left(\left(n^{-1}\ln\ln n\right)^{(1+\alpha)/2}\right). \end{aligned}$$

$C_2 > 0$ is a constant. Part b) of the lemma is now a consequence of (25), (26), and (29). ⊡

In the remaining part of this section we prove the result concerning a.s. convergence rate and asymptotic normality. Here we have

$$\hat{\rho}_{Rn} - \rho_R = \frac{A\left(\hat{B}_n - B\right) - B\left(\hat{A}_n - A\right)}{A\hat{A}_n}. \tag{30}$$

Proof of Theorem 4.2: Define

$$\bar{B}_n = \frac{1}{n}\sum_{i=1}^{n}\varphi(\bar{F}(\bar{X}_i))\left(1 - 2G(Y_i)\right).$$

An application of the law of iterated logarithm leads to

$$\bar{B}_n - B = O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \quad a.s. \tag{31}$$

Using Lemma 6.5, we conclude

$$\begin{aligned} \left|\hat{B}_n - \bar{B}_n\right| &\leq \frac{1}{n}\sum_{i=1}^{n}\left(\left|\varphi(\bar{F}_n(\bar{X}_i)) - \varphi(\bar{F}(\bar{X}_i))\right|(1 - 2G_n(Y_i)) \right. \\ &\qquad \left. + 2\varphi(\bar{F}(\bar{X}_i))\left|G_n(Y_i) - G(Y_i)\right|\right) \\ &= O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \quad a.s. \end{aligned}$$

which proves the Theorem in view of Lemma 6.7a), (30) and (31).

PROOF OF THEOREM 4.3: By (30) and Lemma 6.7b), we have

$$\sqrt{n}\left(\hat{B}_n A - B\hat{A}_n\right) = \sqrt{n}\left(A\,(B_{1n}+B_{2n}) - B\left(\bar{A}_n + \breve{A}_n\right)\right) + o(n^{-1/2}) \quad a.s., \tag{32}$$

where

$$\begin{aligned} B_{1n} &= \frac{2}{n}\sum_{i=1}^{n}\left(\varphi(\bar{F}_n(\bar{X}_i)) - \varphi(\bar{F}(\bar{X}_i))\right)(G(Y_i) - G_n(Y_i)), \\ B_{2n} &= \frac{1}{n}\sum_{i=1}^{n}\Big(\left(\varphi(\bar{F}_n(\bar{X}_i)) - \varphi(\bar{F}(\bar{X}_i))\right)(1 - 2G(Y_i)) \\ &\qquad + \varphi(\bar{F}(\bar{X}_i))\,(1 - 2G_n(Y_i)) - B\Big). \end{aligned}$$

Applying Lemma 6.5, we obtain

$$\begin{aligned} |B_{1n}| &\le 2\sup_{x\in\mathbb{R}^d}\left|\varphi(\bar{F}_n(x)) - \varphi(\bar{F}(x))\right| \sup_{y\in\mathbb{R}}|G_n(y) - G(y)| \\ &= O\left(\frac{\ln\ln n}{n}\right) = o(n^{-1/2}) \quad a.s. \end{aligned} \tag{33}$$

Let $\varphi_j(u) := \frac{\partial}{\partial u_j}\varphi(u)$. By assumption, functions φ_j are Hölder-continuous. Moreover, by Taylor expansion, we have

$$\begin{aligned} B_{2n} &= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{l=1}^{d}\varphi_l(\bar{F}(\bar{X}_i))\left(F_{ln}(X_i^{(l)}) - F_l(X_i^{(l)})\right)(1 - 2G(Y_i)) \right. \\ &\qquad \left. + \varphi(\bar{F}(\bar{X}_i))\,(1 - 2G_n(Y_i)) - B\right) + R_n \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_1((\bar{X}_i, Y_i), (\bar{X}_j, Y_j)) + R_n, \end{aligned}$$

where

$$\begin{aligned} \Lambda_1((x,y),(\bar{x},\bar{y})) &= \sum_{l=1}^{d}\varphi_l(\bar{F}(x))\,(\mathbf{1}\{\bar{x}_l \le x_l\} - F_l(x_l))\,(1 - 2G(y)) \\ &\quad + \varphi(\bar{F}(x))\,(1 - 2\cdot\mathbf{1}\{\bar{y}\le y\}) - B \end{aligned}$$

for $x, \bar{x} \in \mathbb{R}^d, y, \bar{y} \in \mathbb{R}$,

$$\begin{aligned} |R_n| &\le O(\frac{1}{n})\sum_{i=1}^{n}\sum_{l=1}^{d}\left|F_{jn}(X_i^{(j)}) - F_j(X_i^{(j)})\right|^{(1+\alpha)/2} \\ &= O\left(\left(\frac{\ln\ln n}{n}\right)^{(1+\alpha)/2}\right) = o(n^{-1/2}). \end{aligned}$$

Here $\alpha \in (0,1]$ is the Hölder exponent. Further

$$\bar{A}_n + \breve{A}_n = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_2((\bar{X}_i, Y_i), (\bar{X}_j, Y_j)),$$

where

$$\Lambda_2((x,y),(\bar{x},\bar{y})) = \Lambda(-\varphi(\bar{F}(x)) - \int_{-1}^{1}\Lambda(z)\mathrm{d}K(z) + \sum_{l=1}^{d}\bar{\varphi}_l(\bar{F}(x))\left(\mathbf{1}\left\{\bar{x}_l \le x_l\right\} - F_l(x_l)\right).$$

Now we are in a position to show asymptotic normality of $\hat{\rho}_{Rn}$. In view of (30), (32) and (33), we have

$$\sqrt{n}\left(\hat{\rho}_{Rn} - \rho_R\right) = A^{-1}A_n^{-1}B_n^* + o_{\mathbb{P}}(1), \quad \text{where}$$

$$B_n^* = n^{-3/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(A\Lambda_1((\bar{X}_i, Y_i), (\bar{X}_j, Y_j)) - B\Lambda_2((\bar{X}_i, Y_i), (\bar{X}_j, Y_j))\right).$$

Let $\Lambda_0 = A\Lambda_1 - B\Lambda_2$. Further

$$B_n^* = n^{-3/2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\Lambda_0((\bar{X}_i, Y_i), (\bar{X}_j, Y_j)) + \Lambda_0((\bar{X}_j, Y_j), (\bar{X}_i, Y_i))\right) + o_{\mathbb{P}}(1).$$

We deduce

$$\begin{aligned}
&\mathbb{E}\Lambda_0((X_1, Y_1), (\bar{x}, \bar{y})) \\
= \;& -B\int_{\mathbb{R}^d}\sum_{j=1}^{d}\bar{\varphi}_j(\bar{F}(x))\left(\mathbf{1}\left\{\bar{x}_j \le x_j\right\} - F_j(x_j)\right)\mathrm{d}F(x) \\
& +A\left(\int_{\mathbb{R}^{d+1}}\sum_{j=1}^{d}\varphi_j(\bar{F}(x))\left(\mathbf{1}\left\{\bar{x}_j \le x_j\right\} - F_j(x_j)\right)(1 - 2G(y))\,\mathrm{d}H(x,y)\right. \\
& \left. +2\int_{\mathbb{R}^{d+1}}\varphi(\bar{F}(x))\left(G(y) - \mathbf{1}\left\{\bar{y} \le y\right\}\right)\mathrm{d}H(x,y)\right),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\Lambda_0((x,y),(X_1, Y_1)) \;=\; & -B\left(\Lambda(-\varphi(\bar{F}(x))) - \int_{-1}^{1}\Lambda(z)\mathrm{d}K(z)\right) \\
& +A\left(\varphi(\bar{F}(x))\left(1 - 2G(y)\right) - B\right).
\end{aligned}$$

Now we apply the central limit theorem for U-statistics (see Theorem 5.5.1A in Serfling (1980)) to obtain the theorem. ⊡

7. Reference

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